1. INTRODUCTION

A measured quantity has no useful physical significance unless the associated uncertainty or ‘error’ is specified. Used in this booklet the word ‘error’ has a very specific meaning. It does not mean that a mistake has been made. ‘Error’ is a measure of the accuracy of the result. It can provide significant information about the experiment and the result obtained. When a physical quantity is measured, the value obtained is not expected to be equal to the true value. It is important to specify the error associated to indicate how close the result is to the true value. For example, we might measure the acceleration due to gravity g using a simple pendulum and give the result as $g = (9.6 \pm 0.3)$ m s^{-2}. It means that the acceleration due to gravity is somewhere between 9.3 and 9.9 m s^{-2}.

Estimates of errors are important because significant conclusions cannot be drawn from the experimental results without them. For example, the Danish astronomer Tycho Brahe (1546-1601) recorded the planetary positions to an accuracy of better than $1^{o}/360$. These details formed the basis of future developments in the theory of planetary motion. Any new theory of planetary motion had to fit Brahe’s observations. The German astronomer Johann Kepler (1571-1630) was involved in the study of the orbit of Mars. He assumed that the planets moved in circular orbits with constant speed and computed the positions of the planet Mars. Using the data accumulated by Brahe, he tried to fit the planet Mars into a circular orbit. He found that his computed results differed from Brahe’s data by $8^{o}/360$. This difference, though small, lay outside the error of Brahe’s data. Eventually Kepler had to reject the assumption of circular motion and realised that it was an ellipse with the Sun at one focus. This led to the Kepler’s first law of planetary motion. If Brahe’s error was greater than $8^{o}/360$ or unknown, the discrepancy between theory and experiment would not be significant.

It is important to students to analyse the uncertainties associated with their results so that the precision of the final result of their experiments is appropriate. This booklet is intended to help students know:

a. how to use the significant figures,
b. how to locate the sources of errors,
c. how to estimate the magnitude of each error, and
d. how to combine the individual errors into the final error.

2. SIGNIFICANT FIGURES

The number of significant figures reflects the accuracy of a measurement. If the mass of an object is estimated to lie somewhere between 9.235 g and 9.245 g, then the result would be quoted as 9.24 g. Such measurement is said to be given to 3 significant figures. The same result could be quoted as 0.00924 kg so that the accuracy of the determination is still to 3 significant figures. If the result is expressed in milligrams, then a nought is added to maintain the accuracy to 3 significant figures; 9240 mg. But now there is no way of telling that the nought added is to indicate the position of the decimal point, or the measurement has been made to an accuracy of 4 significant figures. To remove any ambiguity it is best

i) to quote an error associated with the measured value, that is (9240 ± 5) mg, or

ii) to express the result in scientific notation, that is, 9.24×10^3 mg and 9.240×10^3 mg to indicate 3 and 4 significant figures respectively.

The number of significant figures used to express a measurement must be <u>consistent with the accuracy of the measurement</u>. It is thus important to round-off the value to the number of significant figures that can be justified. The next significant figure is rounded down if it is less than 5, and rounded up if it is 5 or more.

Some points to note:

i) Round-off the final result according to its error. For example, it is convenient to round-off all readings taken to the last digit calibrated on instruments.

ii) It is important to round-off the final result to the number of significant figures that can be justified.

Example 1:

The lengths of 5 rods are 1.36 cm, 16.72 cm, 5 cm, 0.89 cm, 9.3 cm, what is the total length of the rods when placed in a straight end to end?

The sum of the lengths is:

```
      1.36
     16.72   ← doubtful figure
      5
      0.89
   +  9.3      meaningless figures
   ---------
Total = 33.27
```

the first column that contains a doubtful figure

As the length of the third rod is known only to the nearest centimetre, so the sum is 33 cm, i.e. the accuracy of the final result is governed by the accuracy of the quantity which is measured least accurately. In addition or subtraction, the sum may have more significant figures than some of the original numbers. In this example, note that 5 has only one significant figure, but the sum has two.

In carrying out computations, some meaningless digits may arise from calculations. Such digits which are not significant should be dropped. This is to avoid drawing wrong conclusions because too many digits may imply an accuracy better than the actual measurement. The following rounding rules may be applied:

i) In addition and subtraction, DO NOT carry the result beyond the first column from the left that contains a doubtful figure. The result is rounded off to this column and all digits to the right are dropped.

ii) In multiplication and division, keep the number of significant figures of the final result the same as the number of significant figures in the least precise of the quantities, where 'least precise' means having the lowest number of significant figures.

Example 2:

i) $72.56 + 612 = 685$

$$\begin{array}{r} 72\,.56 \\ +\ 61\mathbf{2} \\ \hline 684\,.56 \\ \downarrow \\ 685 \end{array}$$

doubtful figure

ii)

$$\begin{array}{r} 7\,.\mathbf{3} \\ +\ 2\,.\mathbf{5} \\ \hline 9\,.8 \end{array} \qquad \begin{array}{r} 5\,.347 \\ +\ \mathbf{1} \\ \hline 6 \end{array} \qquad \begin{array}{r} 5\,.347 \\ +\ 1\,.\mathbf{3} \\ \hline 6\,.6 \end{array} \qquad \begin{array}{r} 5\,.347 \\ +\ 0\,.00\mathbf{1} \\ \hline 5\,.348 \end{array} \qquad \begin{array}{r} 3\,.000 \\ +\ 0\,.0\mathbf{2} \\ \hline 3\,.02 \end{array}$$

doubtful figures

iii)

$$\begin{array}{r} \mathbf{2}00 \\ -\quad 3 \\ \hline 200 \end{array} \qquad\qquad \begin{array}{r} 20\mathbf{0} \\ -\quad 3 \\ \hline 197 \end{array}$$

eg. Number of students in a lecture theatre is estimated to be 200. The estimate will not be changed if 3 students leave

e.g. If a student has two \$ 100 note in his pocket and spends \$ 3 then he has \$ 197 left.

Example 3:

Use the following data to calculate the combined weight of two ice-dancers, Mary and John.

mass of Mary $m_M = 45.3$ kg
mass of John $m_J = 50.6$ kg
acceleration due to gravity at the surface of the earth $g = 9.8$ m s^{-2}

The combined weight $= (m_M + m_J)\,g$ least precise quantity

$= (45.3 + 50.6) \times \mathbf{9.8}$

$= 95.9 \times 9.8$

~~= 939.82 N~~ (calculator display)

$= 9.4 \times 10^2$ N

The final result is corrected to 2 significant figures as in *g* which has the least number of significant figures. Note that accuracy cannot be improved by a string of meaningless digits arising from multiplication. Some students tend to copy all the digits which the calculator displays. This should be discouraged.

Example 4:

A toy car of mass 1.22 kg moves on a horizontal ground with speed 3.2 m s^{-1}.

$$\begin{aligned}\text{kinetic energy of the toy car} &= \frac{1}{2}mv^2 \\ &= \frac{1}{2}(1.22)(3.2)^2 \text{ J} \\ &= \cancel{6.2464 \text{ J}} \text{ (calculator display)} \\ &= 6.2 \text{ J}\end{aligned}$$

The final result is corrected to 2 significant figures because the speed of the toy car is known only to 2 significant figures.

Example 5:

i) $\pi\,(10.4)^2 = 3.40 \times 10^2$

$\pi \times (\mathbf{10.4})^2 = 339.7946614$ (calculator display)

ii)

error free	3 sig. fig.		3.40×10^2
4.6	4.6123	$\dfrac{49}{\mathbf{6}}$	$\dfrac{\mathbf{49}}{\mathbf{6.1}}$
× **3.9**	× **3.9**		
17.94	17.98797	= 8.1666667	= 8.0327869
= 18	= 18	= 8	= 8.0

In multiplication and division, the product or quotient cannot have more significant figures than that of the least accurately known original numbers. Usually, during multiplication and division, it is better to carry an extra significant figure along the intermediate steps, and the final answer is then rounded off appropriately.

3. SOURCES OF ERRORS

(a) Instrumental limitations

All measuring instruments have their limitations. Some instruments may have inaccurate scales and others may not be precise enough for the measurement you wish to make. For example, the measurement of the diameter of a pencil using a metre rule has a greater error than that measured by vernier calipers. For an instrument with a scale and pointer, say a voltmeter, the reading accuracy will be mainly determined by the fineness of the scale divisions. The error in a measurement due to instrumental limitations cannot be reduced just by taking repeated measurements.

(b) Systematic Errors

Systematic errors cause all measurements to be shifted systematically in one direction either larger or smaller than it should be. They cannot be reduced by taking repeated measurements.

Examples include:

- parallax in reading scale (when viewing the scale always from one side)
- a zero error on any scale
- a calibration error
- a background count in a radioactivity experiment
- a stray magnetic field
- an error in metre rules due to thermal contraction, etc.

(c) Random Errors

Random errors result from unknown and unpredicted variations in experimental situations. They may be due to :

- random variations in the quantity being measured
- unintended slight changes of the conditions of the experiment
- random variations in the way that measuring instruments are set up, etc.

Examples include:

- parallax in reading scale (when viewing the scale in different directions)
- unpredicted fluctuation in air temperature or line voltage
- unbiased estimates of measurement readings by the observer
- non-uniformity of diameter of a wire, etc.

The effect of random errors can be reduced and minimized by improving experimental techniques and repeating the measurement a sufficient number of times so that the erroneous readings become statistically insignificant.

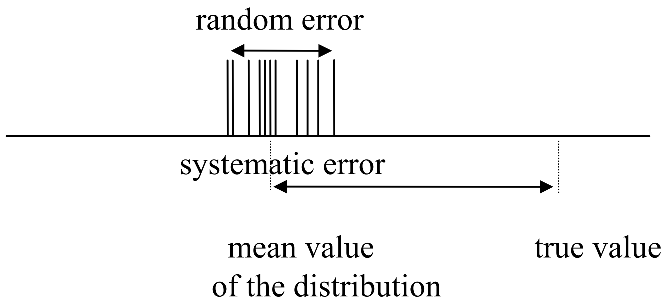


Figure 1. Random and systematic errors

(d) Plain mistakes

These are careless <u>mistakes</u> such as misreading of scale, faulty arithmetic and faulty transcription.

4. TREATMENT OF ERRORS

(a) Instrumental limitations

The scale error is usually taken as half of the smallest division on the scale. For example, the scale of an ordinary thermometer is divided into degrees, so the scale error is 0.5 °C. The room temperature measured would be read as 25.3°C in Figure 2. The "3" in the reading is estimated and is a doubtful figure. The reading is therefore expressed as (25 ± 0.5) °C.

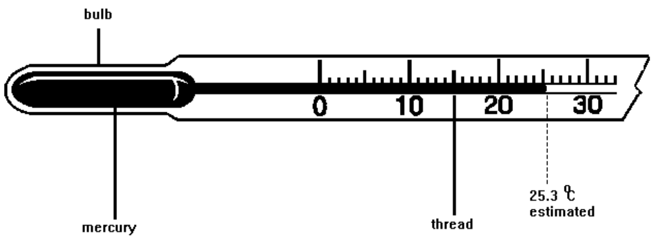
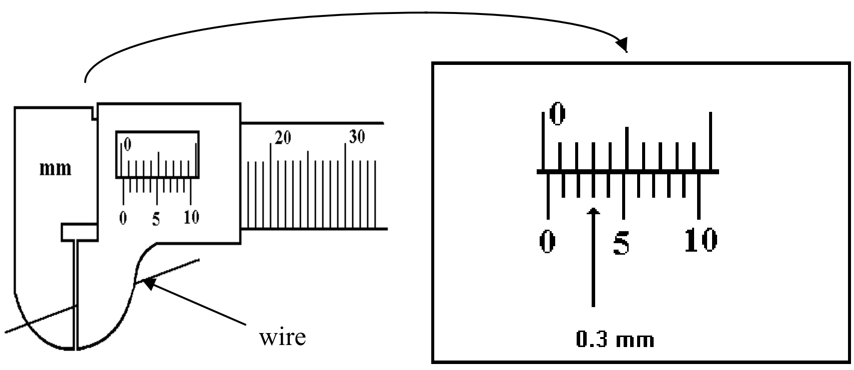


Figure 2. Mercury-in-glass thermometer

Some points to note:

i) Always use an instrument with appropriate scale or sensitivity.

For example:
Assume that the vernier calipers and the micrometer will read zero without an object in place (i.e. there is no zero error). If the diameter of a wire is about 0.3 mm, vernier calipers which would give you a result correct only to the nearest 0.1 mm is not an appropriate instrument to use. The scale error which is half of the smallest division would be 0.05 mm and would give a percentage error of 33.3 % ($2 \times \frac{0.05}{0.3} \times 100\%$). There is a factor of 2 in the calculation because 2 readings have been taken, the zero reading and 0.3 mm (see Figure 3).



The diameter of the wire is (0.3 ± 0.1) mm.

Figure 3. Measuring the diameter of a wire using vernier calipers.

With a micrometer screw gauge, the scale error could be reduced to 0.005 mm. Now, the percentage error is 3 %.

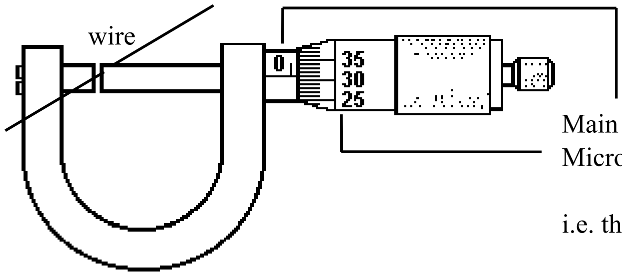


Figure 4. Measuring the diameter of a wire using a micrometer.

ii) Take special care when using very precise instruments.

For example:
Timing a ball-bearing falling through a vertical height of 10 m by a stopwatch could not be more accurate than 0.1 s (reaction time). However, a digital stopwatch would display a reading to the nearest 0.01 s; a time of 1.43 s measured in the experiment should be expressed to (1.4 ± 0.1) s, since the last figure is meaningless when your reaction time could have been as large as 0.1 s.

(b) Systematic Errors

In Figure 1, the systematic errors has shifted the readings so that they are no longer centred about the true value. Under such experimental conditions, the true value of the final result could not be obtained no matter how many readings are taken.

There is *no general rule* for the estimation of systematic errors. Any systematic error that we know about should be corrected and hence eliminated. For example, when using the micrometer screw gauge to take measurements, the zero error should be checked before making a measurement. In some situations, careful individual physical considerations should be taken or the results should be compared with independent measurements. In most cases, the systematic errors are negligible when compared with other errors.

Some common systematic errors and their treatments are given below:

i) If there is a gap between the object being measured and the scale, and the line of sight is not at right angles to the scale, there is a parallax (Figure 5). When readings are taken at different positions, different results would be obtained. Personal bias of an observer, who, for example, always take a reading from the left (position 1) leads to a systematic error of negative value. The parallax error can be avoided by viewing the scale perpendicularly (position 2).

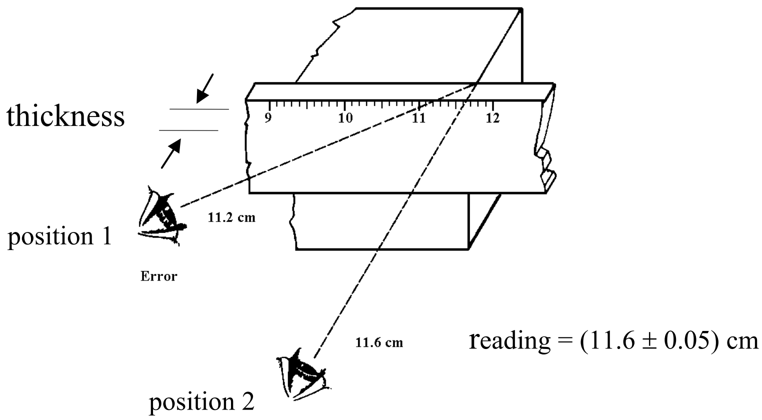
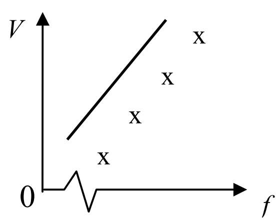


Figure 5 Parallax error in reading a scale

ii) Always check and adjust the zero reading before using an instrument. For example, voltmeters and ammeters need to be checked before use to ensure they read zero when disconnected. Another example is using a micrometer; the reading should be zero without an object in place. If not, add or subtract the zero error from all the readings.

iii) Always take the following effects into consideration:
- background count in radioactivity experiments
- background magnetic field
- heat loss to surrounding
- air resistance
- end correction
- calibration error, etc.

Example 6: (1994 HKALE - PHYSICS -Paper I - Question 37)



A student measures the p.d. V to stop photoelectrons emitted in a photocell illuminated by monochromatic light of various frequencies f. The resulting points when plotted on a V-f graph (as shown) do not lie on the solid line drawn from standard results obtained with a similar photocell. The reason could be

A. the standard results are obtained with light of higher intensity.
B. he has used a voltmeter which has a fixed zero error.
C. he has read the wrong scale on his voltmeter so that his readings always double the actual readings.
D. he has connected the variable d.c. supply with the wrong polarities to the photocell.
E. he has plotted the wavelength of light in place of the frequency on

(*Answer: B*)

the horizontal axis.

(c) Random Errors

These are small fluctuations in the values of repeated measurements and are due to unintended slight changes of conditions of experiment. The readings are just likely to be either too small or too large. Their effect can be reduced by taking a large number of readings.

Random errors are estimated from the extent to which the readings (x_1, x_2, ..., x_n) are spread out about the mean value. We usually take either (i) the **sample standard deviation** or (ii) the **average absolute deviation** of the measurements as measures of the random error to describe the precision of a set of measurements.

i) Sample standard deviation

When a measurement is taken n times, the variability of individual measurement from the sample mean can be measured by the sample standard deviation which is given by

$$\Delta x = \sqrt{\frac{1}{n-1}\sum_{i}^{n}(x_i - \bar{x})^2} = \sigma_{n-1}$$

where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i =$ mean value of $x_1, x_2, \ldots\ldots, x_n$, and

$\sigma_{\text{n-1}}$ is a built - in function in the calculator

$\therefore$ The reading is expressed as $\bar{x} \pm \Delta x$.

ii) Average absolute deviation

$$\Delta x = \frac{1}{n}\sum_{i}^{n}\left|x_i - \bar{x}\right|$$

where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i =$ mean value of $x_1, x_2, \ldots\ldots, x_n$

$\therefore$ The reading is expressed as $\bar{x} \pm \Delta x$.

Example 7:

In five identical experiments to find the magnitude of the acceleration due to gravity, the following results are obtained; 9.5 m s^{-2}, 9.2 m s^{-2}, 9.4 m s^{-2}, 9.6 m s^{-2}, and 9.4 m s^{-2}. Estimate the random error associated with this experiment.

$$\overline{g} = \frac{9.5 + 9.2 + 9.4 + 9.6 + 9.4}{5}$$

$= 9.42$

$= 9.4$ m s^{-2} (to 2 significant figures)

The sample standard deviation $= \sqrt{\frac{1}{n-1}\sum(g_i - \overline{g})^2}$

$= \sigma_{n-1}$

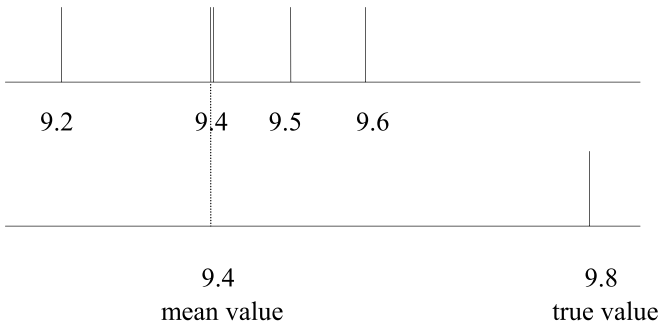
$= 0.148$

$= 0.1$ m s^{-2}

The random error = 0.1 m s^{-2}

$\therefore$ The measured value of g is expressed as (9.4 ± 0.1) m s^{-2}.

If *g* is known to be 9.8 m s^{-2}, then there is a systematic error of negative value.



Example 8:

Diameter of a metal rod may vary at different points because the rod may not be uniform and perfectly circular. The way to minimize the error is to measure the diameter of the rod at least 3 different positions in 2 perpendicular directions. The six readings were recorded as: 17.5 mm, 17.8 mm, 17.6 mm, 17.7 mm, 17.4 mm, 17.8 mm.

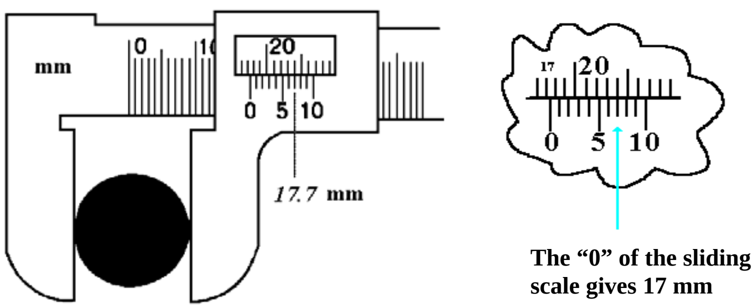


Figure 6 Measuring the diameter of a metal rod using vernier calipers.

(i) What is the mean value of the readings, the sample standard deviation and the average absolute deviation of the readings from the mean?

The mean value $\bar{x} = \dfrac{17.5 + 17.8 + 17.6 + 17.7 + 17.4 + 17.8}{6}$

$= \cancel{17.633333}$

$= 17.6$ mm (3 sig. fig.)

Sample standard deviation $= \sqrt{\dfrac{1}{\text{n - 1}}\sum(x_i - \bar{x})^2}$

$= \sigma_{n-1}$

$= \cancel{0.16}$

$= 0.2$ mm

The average absolute deviation

$= \dfrac{1}{n}\sum|x_i - \bar{x}|$

$= \dfrac{|(17.5 - \bar{x})| + |(17.8 - \bar{x})| + \ldots\ldots\ldots + |(17.4 - \bar{x})| + |(17.8 - \bar{x})|}{6}$

$= \dfrac{0.8}{6}$

$= \cancel{0.133}$

$= 0.1$ mm

[N.B. The round-down value of $\bar{x}$ (17.6 mm) should not be used in the intermediate steps of the calculation. Use all digits displayed on the calculator throughout the calculation process.]

(ii) Write down the diameter of the rod [in the form of ($x \pm \Delta x$) mm].

Diameter of the rod = (17.6 ± 0.2) mm

[N.B. The sample standard deviation is Δx = 0.2 mm which is greater than the uncertainty in single measurement (± 0.1 mm).]

(iii) If the vernier calipers read + 1.0 mm without an object in place, what is the diameter of the rod?

the corrected reading for the object diameter
= [(17.7 ± 0.1) - 1.0] mm
= (16.7 ± 0.1) mm

(d) Plain mistakes

Such mistakes should be avoided by careful work and checks.

5. ESTIMATION OF ERRORS

(a) Absolute, fractional and percentage error

Suppose a reading x is obtained of a quantity whose true value is X,

1. the absolute error $\Delta X = |X - x|$
2. the fractional error $= \left|\dfrac{\Delta X}{X}\right| \approx \left|\dfrac{\Delta X}{x}\right|$
3. the percentage error $= \left|\dfrac{\Delta X}{X}\right| \cdot 100\% \approx \left|\dfrac{\Delta X}{x}\right| \cdot 100\%$

(b) Error in a single quantity

In an experiment where a setting is required (e.g. locating the image of an object in an optics experiment), the true position can be located by *bracketing techniques*. For example in Figure 7, if a sharp image can be obtained when the screen is placed from 30.0 cm to 32.0 cm from a lens, we would write the image distance v = (31 ± 1) cm.

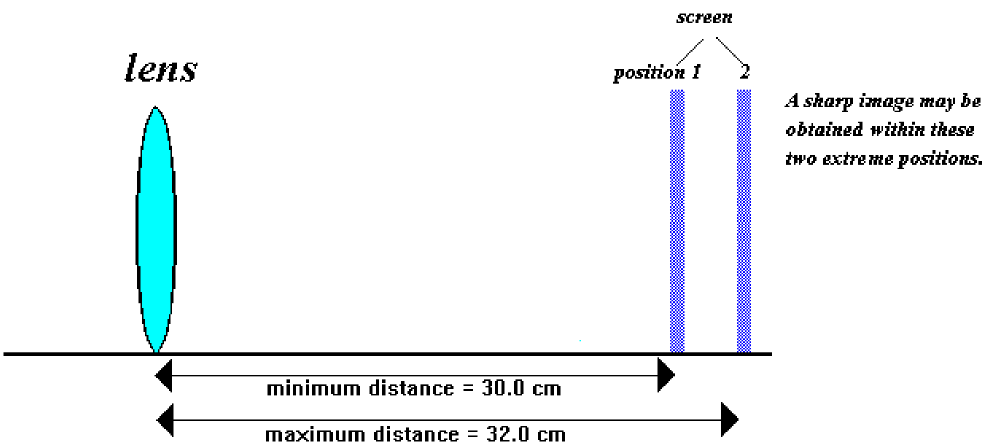


Figure 7 Measurement of image distance from the lens

When a scale has to be read, there will be a <u>scale uncertainty</u>. If a metre rule is used to measure a distance v of 31.0 cm, we would write v = (31.0 ± 0.1) cm, since the smallest division on a metre rule is 0.1 cm.

Clearly, the error in locating the image is greater, therefore the maximum error is 1 cm. The image distance v = (31 ± 1) cm .

Example 9:

The wavelength of sodium light is measured using a diffraction grating which is mounted on the spectrometer table. A single measurement may give a value of (588 $\pm$ 1) nm. Taking more measurements could reduce the random errors and give the following results:

λ/nm	587	589	588	591	588	587	589	590	592	590

$$\bar{\lambda} = \frac{\sum_{i=1}^{10} \lambda_i}{10}$$

$$= \frac{587+589+588+591+588+587+589+590+592+590}{10}$$

$$= \underline{589.1}$$

$$= 589 \text{ nm (3 sig. fig.)}$$

Sample standard deviation $\Delta\lambda$ = 2 nm

The mean value of the above 10 readings is $\bar{\lambda} = 589$ nm. The sample standard deviation of the readings is $\Delta\lambda$ = 2 nm which is greater than the uncertainty in single measurement ($\pm$ 1 nm). Since the wavelength is determined to the nearest nm, it is not appropriate to include the figures after the decimal point. Thus, the wavelength of sodium light measured is expressed as $\bar{\lambda}$ = (589 $\pm$ 2) nm

(c) Combining errors

1. Sum and difference

$Z = A + B$ or $Z = A - B$ where A and B are independent.
For simplicity, the uncertainty in Z, ΔZ can be taken as the sum of individual absolute errors[⚐] :
$\Delta Z = |\Delta A| + |\Delta B|$
This represents the maximum error in Z, so the errors are always added.

Example 10:

If B = (15 $\pm$ 2) cm is subtracted from A = (76 $\pm$ 3) cm, then

$Z = A - B = 76 - 15 = 61$ cm

but the maximum error ΔZ = 3 + 2 = 5 cm i.e. Z = (61 $\pm$ 5) cm.

[⚐] Since the errors are assumed to be independent, it is unlikely that both errors will have their maximum positive value or their maximum negative value at the same time. There may be a partial cancellation, so the best estimate of the error in Z, ΔZ is the square root of the sum of squares of the individual errors:

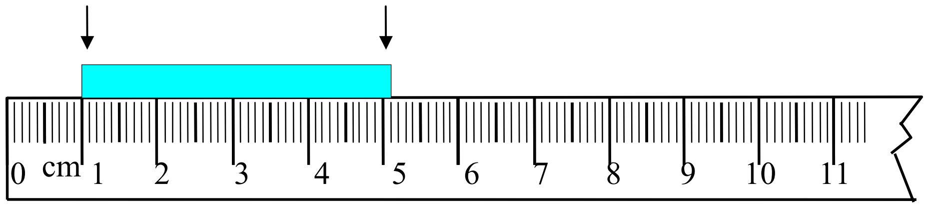
$$\Delta Z_s = \sqrt{|\Delta A|^2 + |\Delta B|^2}$$

But, for simplicity, students are recommended to use the **maximum error** *at sixth form level.*

Example 11:

Use a metre rule to measure the length of an object,

$x_1 = (1.0 \pm 0.05)$ cm $\qquad$ $x_2 = (5.1 \pm 0.05)$ cm



Object length = $x_2 - x_1$ = 5.1 - 1.0
= 4.1 cm

but the maximum error = 0.05 + 0.05 cm = 0.1 cm

i.e. object length = (4.1 ± 0.1) cm

It is a good practice to use the centre of the ruler to measure the length of the object to avoid the zero error at the end which might have been worn-out.

Example 12: (1989 HKALE - PHYSICS -Paper I - Question 1)

A micrometer screw gauge is used to measure the diameter of a piece of wire. The following readings were obtained:

mean zero reading $\quad$ - 0.05 ± 0.02 mm, and
mean apparent diameter $\quad$ + 1.05 ± 0.02 mm.

The diameter of the wire should be written as

A. 1.00 ± 0.02 mm.
B. 1.00 ± 0.04 mm.
C. 1.10 ± 0.00 mm.
D. 1.10 ± 0.02 mm.
E. 1.10 ± 0.04 mm.

(*Answer: E*)

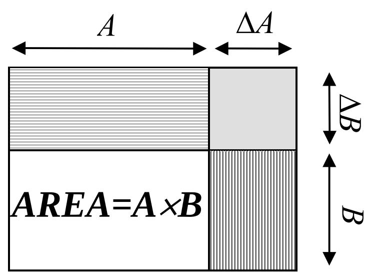
2. Product and Quotient

$Z = A \cdot B \quad$ or $\quad Z = \dfrac{A}{B} \qquad$ where A and B are independent.

To calculate the uncertainty of Z, fractional or percentage errors are used. The maximum fractional error in Z ($\Delta Z/Z$), is the sum of fractional errors in A ($\Delta A/A$) and in B ($\Delta B/B$):

$$\left|\frac{\Delta Z}{Z}\right| = \left|\frac{\Delta A}{A}\right| + \left|\frac{\Delta B}{B}\right|$$

Justification for the rule for compounding errors in multiplication can be obtained algebraically. Suppose that an area with true length A and true width B is measured to have values A + ΔA and B + ΔB, where ΔA and ΔB are the absolute error.



Calculated area = $(A + \Delta A)\cdot(B + \Delta B) = A\cdot B + A\cdot\Delta B + B\cdot\Delta A + \Delta A\cdot\Delta B$

Error in area = calculated area - true area where true area = $A\cdot B$
= $A\cdot\Delta B + B\cdot\Delta A + \Delta A\cdot\Delta B$

Maximum error in area = $|A\cdot\Delta B| + |B\cdot\Delta A| + |\Delta A\cdot\Delta B|$

$$\text{Fractional error in area} = \frac{|A\cdot\Delta B| + |B\cdot\Delta A| + |\Delta A\cdot\Delta B|}{A\cdot B}$$

$$= \left|\frac{\Delta B}{B}\right| + \left|\frac{\Delta A}{A}\right| + \left|\frac{\Delta A\cdot\Delta B}{A\cdot B}\right|$$

$$\approx \left|\frac{\Delta A}{A}\right| + \left|\frac{\Delta B}{B}\right|$$

= fractional error in A + fractional error in B

Example 13:

In the measurement of the wavelength of a sound from its velocity and frequency using $v = f\cdot\lambda$:

v = (330±20) m s^{-1} , $\Delta v / v$ = 20/330 = 0.061 (or 6.1%)
f = (512±10) Hz , $\Delta f / f$ = 10/512 = 0.020 (or 2.0%)
λ = 330/512 = 0.645 m (3 sig. fig.)

The maximum fractional error in λ is
$\Delta\lambda / \lambda$ = 0.061 + 0.020 = 0.081 (or 8.1%), and
$\Delta\lambda$ = 0.645 × 0.081 = 0.052 m

Therefore the value of wavelength calculated from the above data is
λ = (0.65 ± 0.05) m.

3. Power

$Z = k{\cdot}A^n$ where k and n are constants assumed to be error free

The fractional error in Z is n times the fractional error in A:

$$\left|\frac{\Delta Z}{Z}\right| = n \cdot \left|\frac{\Delta A}{A}\right|$$

Example 14: (1993 HKALE - PHYSICS -Paper I - Question 50)

A parallel-plate capacitor is formed by two square metal plates. To determine the capacitance C, a student measures the length of side l and separation d of the plates.
If the maximum percentage error of l = 5%
the maximum percentage error of d = 3%,
then the maximum percentage error for C will be

A. 7%.
B. 8%.
C. 13%.
D. 22%.
E. 28%

(*Answer: C*)

Example 15: (1995 HKALE - PHYSICS -Paper II - Question 45)

In an experiment to measure the density of steel, a steel sphere was used. The following measurements were obtained:

Mass of the sphere = 530 mg ± 1 mg
Diameter of the sphere = 0.51 cm ± 0.01 cm

Estimate the percentage error in the calculated value of the density of steel.

A. 1 %
B. 2 %
C. 4 %
D. 6 %
E. 8 %.

(*Answer: D*)

***Example 16**:*

Find the maximum possible error associated in the measurement of the kinetic energy (E) of an object travelling at velocity v if the mass m = (3.5 ± 0.1) kg and v = (20 ± 1) m s^{-1}.

$$E = \frac{1}{2}mv^2 = \frac{1}{2}(3.5)\cdot(20)^2 = 700 \text{ J}$$

$$\left|\frac{\Delta E}{E}\right| = \left|\frac{\Delta m}{m}\right| + 2 \times \left|\frac{\Delta v}{v}\right| = \frac{0.1}{3.5} + \frac{2 \times 1}{20} = \frac{9}{70}$$

$$i.e.\quad \Delta E = 700 \times \frac{9}{70} = 90 \text{ J}$$

and $E = (700 \pm 90)$ J

Example 17: (1990 HKALE - PHYSICS -Paper I - Question 1)

The formula $T^2 = \dfrac{4\pi^2 l}{g}$ is used to calculate the acceleration due to gravity g.

If the maximum percentage error of l = 2%
the maximum percentage error of T = 5%,
then the maximum percentage error for g will be

A. 3%.
B. 8%.
C. 12%.
D. 23%.
E. 27%.

(*Answer: C*)

Example 18: (1992 HKALE - PHYSICS -Paper I - Question 50)

To determine the area of cross-section of a metal wire, a student measures its diameter and obtains a value of 0.20 mm, subject to an error of ± 0.02 mm. Which of the following is the most appropriate way of expressing the result?

A. 0.03 ± 0.01 mm^2
B. 0.031 ± 0.003 mm^2
C. 0.031 ± 0.006 mm^2
D. 0.0314 ± 0.0031 mm^2
E. 0.0314 ± 0.0063 mm^2

(*Answer: C*)

Example 19: (1996 HKALE - PHYSICS -Paper II - Question 45)

The period of oscillation, T, of a simple pendulum is related to its length, l, by the formula $T = 2\pi\sqrt{\dfrac{l}{g}}$. To find experimentally the acceleration of free fall by using a simple pendulum, a student takes the following measurements:

time for 15 oscillations: 14.4 ± 0.2 s
length of the pendulum : 0.229 ± 0.001 m

Which of the following is the most appropriate way of expressing the result ?

A. 9.8 ± 0.2 m s^{-2}
B. 9.8 ± 0.3 m s^{-2}
C. 9.81 ± 0.32 m s^{-2}
D. 9.81 ± 0.179 m s^{-2}
E. 9.810 ± 0.315 m s^{-2}

(*Answer: B*)

4. Other combinations

Some functions, such as sines, cosines and logarithms, may occur in an equation. The fractional errors can still be derived but they are more complicated than those dealt with previously. The simplest way to treat such errors is to calculate the extreme values of the function, and then calculate the maximum deviation of these extreme values from the mean value of the function calculated from the measurements.

Example 20:

In an optics experiment, the wavelength λ of a monochromatic light can be calculated from the formula $\lambda = d \sin \theta$. If $d = (3.30 \pm 0.05) \times 10^{-6}$ m and the measured values of $\theta = 10.1^{\circ}$, 9.9°, 10.3°, and 10.2°, find the value of λ (and its maximum error).

$$\begin{aligned}
\overline{\lambda} &= d \times sin\,\overline{\theta} \\
&= 3.30 \times 10^{-6} \times \sin\left(\frac{10.1^{\circ} + 10.3^{\circ} + 10.2^{\circ} + 9.9^{\circ}}{4}\right) \\
&= 3.30 \times 10^{-6} \times \sin(10.125^{\circ}) \\
&= 5.80 \times 10^{-7} \text{ m}
\end{aligned}$$

As $\sin \theta$ is an increasing function, the extreme values of λ are

$$\lambda_{min} = d_{min} \times sin\,\theta_{min} = 3.25 \times 10^{-6} \times \sin(9.9^{\circ}) = 5.59 \times 10^{-7} \text{ m}$$

$$\lambda_{max} = d_{max} \times sin\,\theta_{max} = 3.35 \times 10^{-6} \times \sin(10.3^{\circ}) = 5.99 \times 10^{-7} \text{ m}$$

$$\lambda_{max} - \overline{\lambda} = 0.19 \times 10^{-7} \text{m}$$

$$\overline{\lambda} - \lambda_{min} = 0.21 \times 10^{-7} \text{m}$$

$$\therefore \text{maximum error} = 0.2 \times 10^{-7} \text{m}$$

$$\overline{\lambda} = (5.8 \pm 0.2) \times 10^{-7} \text{ m}$$

6. Examples from HKALE

(1997 HKALE - PHYSICS -Paper II - Question 45)

In an experiment to determine the period of oscillation, T, of a simple pendulum, the time, t, for a number of complete oscillations is taken. It is found that the time for 30 complete oscillations is 28.7 ± 0.3 s. Which of the following statements is/are correct ?

(1) The reading error in t can be reduced by counting 50 oscillations.
(2) The percentage error in T is the same as that in t.
(3) The period T determined should be quoted as 0.96 ± 0.01 s.

A. (1) only
B. (3) only
C. (1) and (2) only
D. (2) and (3) only
E. (1), (2) and (3) *(Answer:D)*

(1999 HKALE - PHYSICS -Paper 2 - Question 45)

Which of the following will give rise to a systematic error in experimental measurement ?

(1) using a slow-running stop watch in keeping time
(2) ignoring the background counts in radioactive decay measurements
(3) uncertainty in locating the sharpest image position in an optics experiment

A. (1) only
B. (3) only
C. (1) and (2) only
D. (2) and (3) only
E. (1), (2) and (3) *(Answer:C)*

(2001 HKALE - PHYSICS -Paper 2 - Question 36)

In an experiment to determine the Young modulus for a steel wire, a student obtained the following data :

length of steel wire	=	1.96 ± 0.01 m
diameter of steel wire	=	0.61 ± 0.01 mm
mass of the load	=	10.00 ± 0.01 kg
extension	=	3.9 ± 0.1 mm
acceleration of free fall	=	9.8 ± 0.1 m s^{-2}

Which of the following leads to the greatest uncertainty in the calculated value of the Young modulus ?

A. measurement of length
B. measurement of diameter
C. measurement of load
D. measurement of extension
E. assumed value of the acceleration of free fall *(Answer:B)*

(2002 HKALE - PHYSICS -Paper 2 - Question 45)

The diameter of the bore of a capillary tube can be determined by introducing a small quantity of mercury into the capillary. It is possible to measure the length of the mercury thread to within 2% and the mass of the mercury used to within 4%. Assuming negligible error in the density of mercury, the percentage error in the calculated diameter of the capillary bore is at most

A. 2%.
B. 3%.
C. 4%.
D. 6%. (*Answer:B*)

(2003 HKALE - PHYSICS -Paper 2 - Question 13)

Four students, *P, Q, R* and *S*, each made a number of measurements of the acceleration of free fall *g* using the simple pendulum method so as to obtain a mean value. Their results are tabulated as follows :

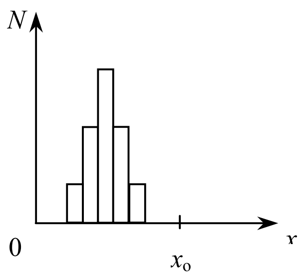
Student	Result g/m s^{-2}
P	9.4 ± 0.8
Q	9.6 ± 0.2
R	9.8 ± 0.3
S	9.9 ± 1.2

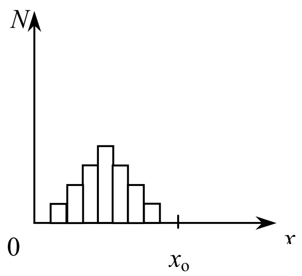
Which student obtained a result that has the largest systematic error ?

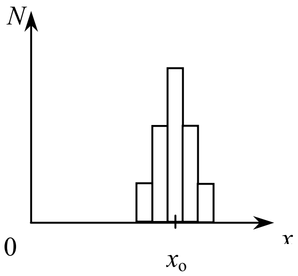
A. *S*
B. *R*
C. *Q*
D. *P* (*Answer:D*)

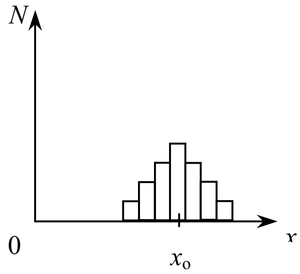
(2005 HKALE - PHYSICS -Paper 2 - Question 25)

A physical quantity x is measured many times. The number of measurements N giving a value of x is plotted against x. The 'true' value of the physical quantity is x_o. Which graph below shows that the measurements have small random errors but a large systematic error? (*Answer:A*)

A.


B.


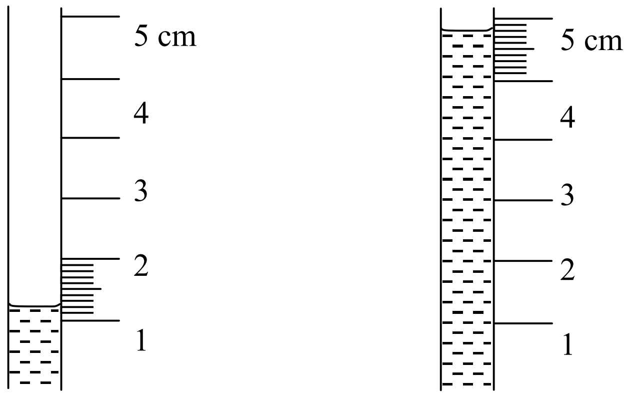
C.


D.


(1999 HKALE - PHYSICS -Paper 1 - Question 1)

A student tries to measure the density of steel. He puts 20 identical steel ball bearings into a measuring cylinder half filled with water. Figure 1.1 shows the readings of the water level before and after placing the bearings into the cylinder. The ruler used is graduated in mm.

Figure 1.1



Without ball bearings With ball bearings

(a) The rise in water level should be recorded as _______ ± ______ (unit: ______).

(b) State and explain **one** precaution when placing the bearings into the cylinder.

(c) The internal diameter of the measuring cylinder is found to be 9.40 $\pm$ 0.05 mm and the total mass of the ball bearings is 25.00 $\pm$ 0.01 g.

(i) Name the instrument used for measuring the internal diameter of the cylinder.

(ii) Find the density of the steel ball bearings.

(iii) Give the percentage uncertainty in your calculated value in (ii). Show your working.

(d) State another method of measuring the density of the ball bearings with a smaller percentage uncertainty. Explain briefly.

(2003 HKALE - PHYSICS -Paper 1 - Question 9)

A student performs the following experiments to find the density and Young modulus of a roll of uniform fishing line.

(a) The mass of the roll of fishing line is weighed by using an electronic balance. Its volume is measured by putting it into a measuring cylinder containing water. The results are as follows :

mass of fishing line m = 2.756 g
initial water level l_1 = 5.0 ml
final water level l_2 = 7.6 ml

(i) State one precaution in measuring the volume of the roll of fishing line.

(ii) Calculate the density ρ of the roll of fishing line to 2 significant figures.

(iii) The smallest divisions of the electronic balance and the measuring cylinder are 0.001 g and 0.1 ml respectively. Estimate the maximum percentage error in ρ

The End

一、簡介

如果沒有說明測量值的不確定值 (即誤差)，則該測量值便沒有物理意義。本文中「誤差」一詞有其特定意義，它並非指任何過錯，而是指測量值的準確度。誤差蘊含著有關實驗及其結果的重要信息。量度一個物理量時，所得的不一定是真值，誤差顯示所得值接近真值的程度。例如以單擺測量某地重力加速度 g，得出結果爲 (9.6 ± 0.3) m s^{-2}，則表示該處的重力加速度在 9.3 至 9.9 m s^{-2} 之間。

不估算誤差，就不能從實驗得出有意義的結論。物理學史上有一個很好的例子。丹麥天文學家第谷 (Tycho Brahe, 1546-1601) 記錄行星位置準確到十秒(秒爲角度單位，符號爲「″」，三千六百秒爲一度)。這些觀測成爲日後發展行星運動理論的基石，任何新建立的行星運動理論都應符合第谷的觀測結果。及後他的學生德國天文學家開普勒 (Johann Kepler, 1571-1630) 在研究火星軌道時，曾假設行星以勻速率在正圓軌道上運行，計算火星的位置。開普勒將計算結果與第谷累積的數據比較，發覺兩者相差八十秒。此差別雖很小，但已超出第谷所指出的允許誤差。這使開普勒放棄圓周運動的假設，而導出火星軌道爲一個橢圓，這就是開普勒行星運動第一定律。如果第谷所得數據的誤差大於八十秒或未被記錄，理論與觀察的衝突將不會如此顯著。

同學應學會分析結果的不確定值(誤差)，並能以適當的準確度表達其實驗結果。希望本文可以幫助同學掌握以下幾方面:

a. 運用有效數字；
b. 找出誤差來源；
c. 估算每種誤差的大小；及
d. 將個別誤差組合。

二、有效數字

有效數字的數目反映了測量值的準確性。若估算某物體的質量在 9.235 g 至 9.245 g 之間，則結果應寫為 9.24 g。該測量值具有三位有效數字。同一結果亦可以 0.00924 kg 表示，準確度仍為三位有效數字。若結果以毫克表示，則需在個位補零，即 9240 mg，但此寫法亦可能表示測量值具有四位有效數字的準確度。為防止出現類似的含糊情況，最好採取以下做法：

i) 將誤差與測量值一同標出。
例如：(9240 ± 5) mg；或

ii) 以科學記數法表示結果。
例如：9.24×10^3 mg 和 9.240×10^3 mg 分別表示測量值具有三位及四位有效數字。

採用的有效數字<u>需與測量值的準確度一致</u>，額外(超精確)數位的數字應作四捨五入。

注意要點：

i) 根據測量值的誤差而作取捨。
例如：所有數據應讀至儀器標度的最精確刻度。

ii) 將最後結果調至恰當位數的有效數字。

例一　五根棒的長度分別為 1.36 cm、16.72 cm、5 cm、0.89 cm 和 9.3 cm，若將各棒沿一直線頭尾相接擺放，總長度是多少？

解：　　總長度為：

$$
\begin{array}{r}
1.36 \\
16.72 \\
\mathbf{5} \\
0.89 \\
+\quad 9.3 \\
\hline
\text{總和} = 33.\boxed{27}
\end{array}
$$

可疑數

無意義數

自左起第一個含有可疑數字的欄

因為第三根棒的長度只準確到 1 cm，所以總長度為 33 cm，即最後結果的準確度取決於數據中的最低精密度，在加法或減法中，結果的有效數字數目可能比某些原始數據的為多。此例中的第三個數據「5」只有一位有效數字，但其總和則有兩位有效數字。

計算過程中，常衍生無意義的數值。該等數值不應保留，否則所得結果蘊含的準確度比原始數據的準確度為高，為了免生這種謬誤，應採用以下取捨原則：

i) 進行加法和減法運算時，自左起找出第一個含有可疑數字的欄位，再將該位右邊的數字作四捨五入。(參閱例一及例二)

ii) 進行乘法和除法運算時，所得的積和商的有效數字數目，應與最低精密度的原始數據的有效數字數目相等。所謂「最低精密度的數據」就是指有效數字數目最少的數據。(參閱例三至例五)

例二 i) 72.56 + 612 = 685

$$\begin{array}{r} 72.56 \\ +\ 61\mathbf{2} \\ \hline 684.56 \\ \downarrow \\ 685 \end{array}$$

可疑數字

ii)

$$\begin{array}{r} 7.\mathbf{3} \\ +\ 2.\mathbf{5} \\ \hline 9.8 \end{array} \qquad \begin{array}{r} 5.347 \\ +\ \mathbf{1} \\ \hline 6 \end{array} \qquad \begin{array}{r} 5.347 \\ +\ 1.\mathbf{3} \\ \hline 6.6 \end{array} \qquad \begin{array}{r} 5.347 \\ +\ 0.00\mathbf{1} \\ \hline 5.348 \end{array} \qquad \begin{array}{r} 3.000 \\ +\ 0.0\mathbf{2} \\ \hline 3.02 \end{array}$$

可疑數字

iii)

$$\begin{array}{r} \mathbf{2}00 \\ -\ 3 \\ \hline 200 \end{array} \qquad\qquad \begin{array}{r} 20\mathbf{0} \\ -\ 3 \\ \hline 197 \end{array}$$

如：估計約有二百名學生在講室內，當有三名學生離去後，則學生人數仍約為二百名。

如：若一學生的口袋中有兩張 \$100紙幣，用去 \$3後將剩下 \$197。

例三 根據以下的數據計算秀文和伊健的總重量。

秀文的質量 $m_s = 45.3$ kg

伊健的質量 $m_e = 50.6$ kg

地球表面之重力加速度 $g = 9.8$ m s^{-2}

解：總重量 $= (m_s + m_e)\, g$

最低精密度的物理量

$= (45.3 + 50.6) \times \mathbf{9.8}$

$= 95.9 \times 9.8$

$= \cancel{939.82}$ (計算機顯示的數字)

$= 9.4 \times 10^2$ N

數據中 g 的精確度最低，只有兩位有效數字。因此，答案也只需要準確到兩位就可以了。值得注意的是，由乘法運算得出來的一連串數字，並不能提高答案的精密度。部分同學不加思索地把計算機顯示的數字全抄下來的做法，必須加以糾正。

例四　一輛質量爲 1.22 kg 的玩具車，在水平地面以 $3.2\ \text{m s}^{-1}$ 的速率運動。求玩具車的動能。

$$\begin{aligned}\text{玩具車的動能} &= \frac{1}{2}mv^2 \\ &= \frac{1}{2}(1.22)(3.2)^2 \\ &= \cancel{6.2464} \quad (\text{計算機顯示的數字}) \\ &= 6.2\ \text{J}\end{aligned}$$

由於速率($3.2\ \text{m s}^{-1}$)只有兩位有效數字，所以答案只需要準確到兩位有效數字。

例五　i)　$\pi\,(10.4)^2 = 3.40 \times 10^2$

$\pi \times (\mathbf{10.4})^2 = 339.7946614$　(計算機顯示的數字)

無誤差　三位有效數字 $\longrightarrow$ 3.40×10^2

ii)

4.6 × **3.9**	4.6123 × **3.9**	$\frac{49}{\mathbf{6}}$	$\frac{\mathbf{49}}{\mathbf{6.1}}$
17.94	17.98797	= 8.1666667	= 8.0327869
= 18	= 18	= 8	= 8.0

進行乘法和除法運算時，所得的積和商的有效數字數目，不可能多於最低精密度的原始數據的有效數字數目。在乘法和除法的中間步驟，常比最低精確數據的有效數字多保留一位，在得出答案後再作適當的捨入。

三、誤差來源

(甲)　儀器的限制

量度儀器一般都有本身的限制。有些儀器的標度不準確，另一些的精密度則不能達到量度的所需要求。例如，以米尺量度鉛筆直徑的誤差，會比以游標卡尺來量度的爲大。這是由於讀數的精密度，是取決於標度間格的大小。只用重複量度並不會減少由儀器的限制所引起的誤差。

(乙) 系統誤差

系統誤差會導致所得的讀數向某一方向偏移，重複量度並不能減少此類誤差。

例子包括：

- 讀取數據時的視差（習慣由標度的某一方讀取數據）
- 標度的零點誤差
- 校準誤差
- 放射性實驗中的本底輻射計數
- 雜散磁場
- 因米尺的熱脹冷縮而導致的誤差，等等

(丙) 隨機誤差

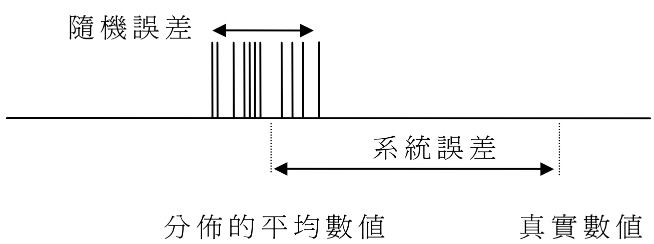
隨機誤差是由實驗時一些不明因素或難以預測的變化所引起的。成因包括：

- 物理量的隨機變化
- 實驗中一些未被察覺的輕微變化
- 在裝置測量儀器時引起不可預測的影響，等等

例子包括：

- 讀取數據時的視差（由不同方位望向標度）
- 空氣溫度或市電的未能預測的波動
- 讀取數據時，觀察者所作的估計
- 導線直徑的非均勻性，等等

透過改善實驗技術和多次重複實驗，可以減少隨機誤差的影響。從統計學的角度來看，多次重複測量，有助減少由個別錯讀帶來的影響。



圖一　隨機和系統誤差

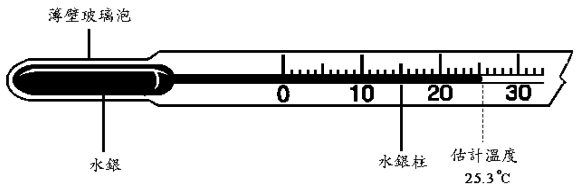
(丁) 人爲錯誤 (Plain mistakes)

人爲錯誤的例子包括錯讀標度，運算及抄寫時的筆誤等。

四、誤差處理

(甲) 儀器的限制

爲求簡易，同學可選取標度最小間隔的一半爲標度誤差。例如：一支普通溫度計的標度若是以最小間隔表示一度，標度誤差就是 0.5 ℃。在圖二中，量度到的室溫是 25.3 ℃，讀數中的“3”是從估計得來的，它是可疑數字，所以應將讀數寫作 (25 ± 0.5)℃。



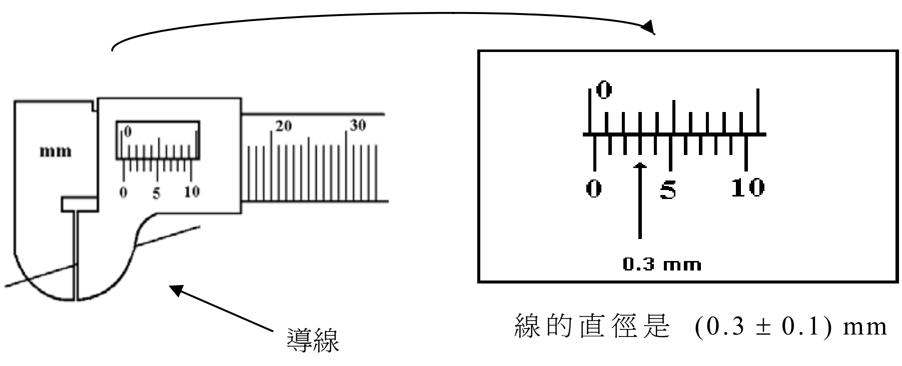
圖二　水銀溫度計

注意事項：

i)　選用有合適標度和靈敏度的儀器

例如：游標卡尺和螺旋測微計均可用作量度物件厚度。但若要量度一條直徑約爲 0.3 mm 的導線，游標卡尺並不合適。因游標卡尺的精密度是 0.1 mm，即標度誤差爲 0.05 mm，所得的百分誤差將爲 33%$\left(2 \times \frac{0.05}{0.3} \times 100\%\right)$。

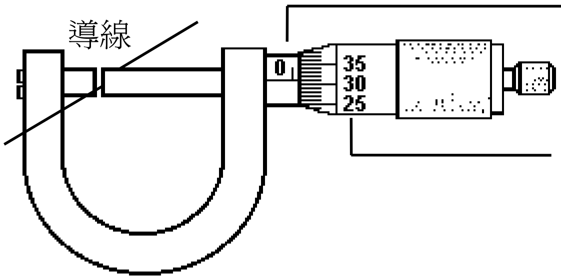
【註：算式中因子“2”是由於要讀取兩端讀數(即零點和 0.3 mm)而來。】



【假設在進行量度前，游標卡尺獲得讀數“0” (即儀器沒有零點誤差)。】

圖三　用游標卡尺量度導線的直徑

若使用螺旋測微計，標度上的誤差則爲 0.005 mm，百分誤差爲 3 %。故此，在這情況下，螺旋測微計較爲適合。



【假設在進行量度前，螺旋測微計獲得讀數“0” (即儀器沒有零點誤差)。】

圖四　用螺旋測微計量度導線的直徑

ii) 使用精密度高的儀器時須留意的地方

如用秒錶量度一個球體下落 10 m 所需的時間，因受反應時間限制，時間讀數的誤差不可能小於 0.1 s。但跳字式計時器的精密度可達 0.01 s。由於反應時間是 0.1 s，那百分之一位數字便沒有意義。故此，如跳字式計時器顯示 1.43 s，結果應寫作（1.4 ± 0.1）s。

(乙) 系統誤差

在圖一中，系統誤差使讀數偏移一方。真實數據不再是所得讀數分佈的中心。在這情況下，無論讀取多少數據都不可能獲得真值。

在估計系統誤差時，沒有通則可依。我們應該把所知的系統誤差逐一糾正或消除。例如：在使用螺旋測微計前，應先檢查是否有零點誤差。在某些情況下，必須考慮個別的物理狀況或比較各項獨立量度的結果。由於系統誤差通常較其他誤差小，因此可忽略不計的。

處理系統誤差的方法包括：

i) 假如被量度的物件與標度有一段距離，而視線又並非垂直於標度，便會產生視差（圖五）。在不同的位置量度，會得到不同的結果。例如，若某觀察者慣於左方(即位置一)讀取數據，將引致負值的系統誤差。垂直地觀察（即位置二）可避免由視差造成的誤差。

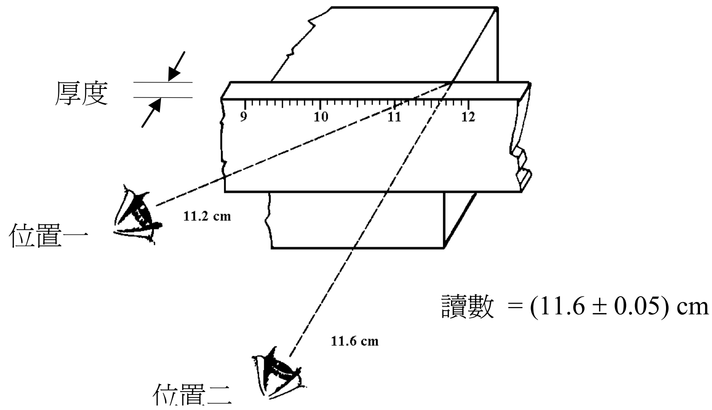


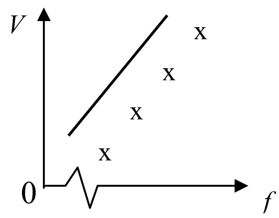
圖 五　　讀 取 數 據 時 的 視 差

ii) 使 用 儀 器 前 ， 必 須 檢 查 和 調 校 讀 數 零 點 。 例 如 ， 在 使 用 伏 特 計 和 電 流 計 時，必 須 先 確 定 在 沒 有 電 流 通 過 的 情 況 下 ， 讀 數 為 零 。 如 使 用 螺 旋 測 微 計 時 ， 在 進 行 量 度 前 ， 讀 數 應 為 零 ， 否 則 應 把 讀 數 加 上 或 減 去 零 點 誤 差 。

iii) 在 收 集 數 據 時 ， 應 考 慮 以 下 的 影 響 ：

- 輻 射 實 驗 中 的 本 底 輻 射 計 數
- 地 球 磁 場 〔 本 底 磁 場 〕
- 散 失 在 環 境 中 的 熱 能
- 空 氣 阻 力
- 端 部 修 正
- 校 準 誤 差 ， 等 等

例六 (1994年　香港高級程度會考　物理科　卷一　第37題)



一學生用不同頻率 f 的單色光照射光電池，並量度用來遏止發出的光電子的電位差 V。當他將結果繪在V-f 表上，所得的點（如圖示）並不落於從類似光電池所得標準結果而繪成的實線上。原因可能是

A. 標準結果所用的光強度較高。
B. 他所用的伏特計有固定的零點誤差。
C. 他使用了伏特計上錯誤的標度讀數，以致他的讀數雙倍於真正讀數。
D. 他把可變直流電源接駁光電池時，將兩極弄錯。
E. 他誤將光的波長作為水平軸上的頻率。　　　　(答案: B)

(丙) 隨機誤差

個別實驗條件的輕微變化，會導致在重複實驗時，數據上出現波動。這些波動所構成的誤差，稱為隨機誤差。讀取大量讀數，有助減小隨機誤差。

隨機誤差的大小取決於讀數 ($x_1, x_2, ..., x_n$) 在平均數值兩旁分佈的寬度。它反影了有關數據的精確度。以下是兩種估算隨機誤差的方法：

i) <u>樣本標準差</u>

經過重複量度後，個別讀數與樣本平均值的差距可由樣本標準差來估算。

$$\Delta x = \sqrt{\frac{1}{n-1}\sum_{i}^{n}(x_i - \bar{x})^2} = \sigma_{n-1}$$

其中 $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$,是 $x_1, x_2, ..., x_n$ 的平均值，及

$\sigma_{\text{n-1}}$ 是計數機中的一項內置功能

∴ 讀數應寫作 $\bar{x} \pm \Delta x$。

ii) <u>平均絕對差</u>

$$\Delta x = \frac{1}{n}\sum_{i}^{n}\left|x_i - \bar{x}\right|$$

其中 $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$,是 $x_1, x_2, ..., x_n$ 的平均值

∴ 讀數應寫作 $\bar{x} \pm \Delta x$。

例七　在求取重力加速度的實驗中，把實驗重複五次，得出以下結果：9.5 m s^{-2}, 9.2 m s^{-2}, 9.4 m s^{-2}, 9.6 m s^{-2} 及 9.4 m s^{-2}。估計實驗結果的隨機誤差。

$$\overline{g} = \frac{9.5 + 9.2 + 9.4 + 9.6 + 9.4}{5}$$

$$\cancel{= 9.42}$$

$$= 9.4 \text{ m s}^{-2} \text{ (兩位有效數字)}$$

$$\text{樣本標準差} = \sqrt{\frac{1}{\text{n - 1}} \sum (g_i - \overline{g})^2}$$

$$= \sigma_{n-1}$$

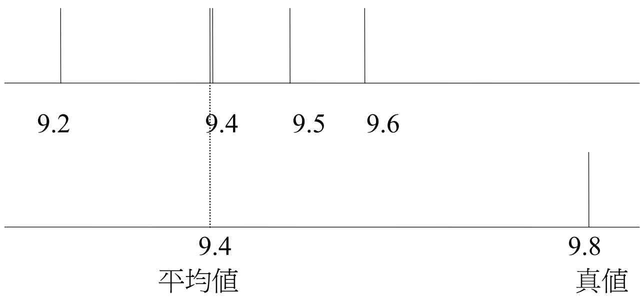
$$\cancel{= 0.148}$$

$$= 0.1 \text{ m s}^{-2}$$

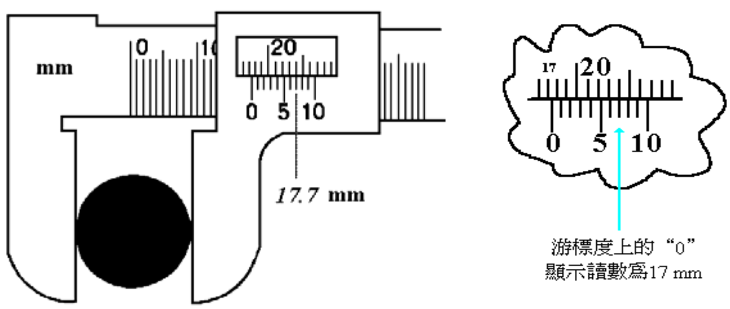
隨機誤差 $= 0.1 \text{ m s}^{-2}$

$\therefore$ 量度所得的 g 應寫作　$(9.4 \pm 0.1) \text{ m s}^{-2}$.

如果 g 的真值是 9.8 m s^{-2}，這表示了實驗結果有負數值的系統誤差。



例八　由於金屬棒的不均勻或其橫切面非正圓，金屬棒的直徑在不同的位置是不同的。因此要減小量度直徑時的誤差，最少應該在三個不同的位置，各沿兩個互相垂直方向量度。如得以下六個讀數：
17.5 mm, 17.8 mm, 17.6 mm, 17.7 mm, 17.4 mm, 17.8 mm.



圖六　使用游標卡尺量度金屬棒的直徑

(i) 求讀數的平均值，樣本標準差和平均絕對差。

$$平均值\ \bar{x} = \frac{17.5+17.8+17.6+17.7+17.4+17.8}{6}$$
$$= \cancel{17.633333}$$
$$= 17.6\ \text{mm}\ (三位有效數字)$$

$$樣本標準差 = \sqrt{\frac{1}{\text{n - 1}}\sum(x_i - \bar{x})^2}$$
$$= \sigma_{n-1}$$
$$= \cancel{0.16}$$
$$= 0.2\ \text{mm}$$

平均絕對差

$$= \frac{1}{n}\sum|x_i - \bar{x}|$$
$$= \frac{|(17.5-\bar{x})| + |(17.8-\bar{x})| + \ldots\ldots\ldots + |(17.4-\bar{x})| + |(17.8-\bar{x})|}{6}$$
$$= \frac{0.8}{6}$$
$$= \cancel{0.133}$$
$$= 0.1\ \text{mm}$$

[注意：在計算過程中，不要使用已四捨五入的平均值 (17.6 mm)，而應採用計算機上顯示的所有數字。]

(ii) 寫出此棒的直徑 [以 $(x \pm \Delta x)$ mm表示。]
棒的直徑 = (17.6 ± 0.2) mm
[注意：樣本標準差 Δx = 0.2 mm 大於任何一次量度的不確定量 (± 0.1 mm)。]

(iii) 進行量度前，若游標卡尺的讀數是+ 1.0 mm，求此棒的直徑。
棒的直徑 = [(17.6 ± 0.1) - 1.0] mm
= (16.6 ± 0.1) mm

(丁) 人為錯誤 (Plain mistakes)

為減少人為錯誤，在做實驗時，應審查各實驗條件並小心處理各實驗步驟。

五、誤差估計

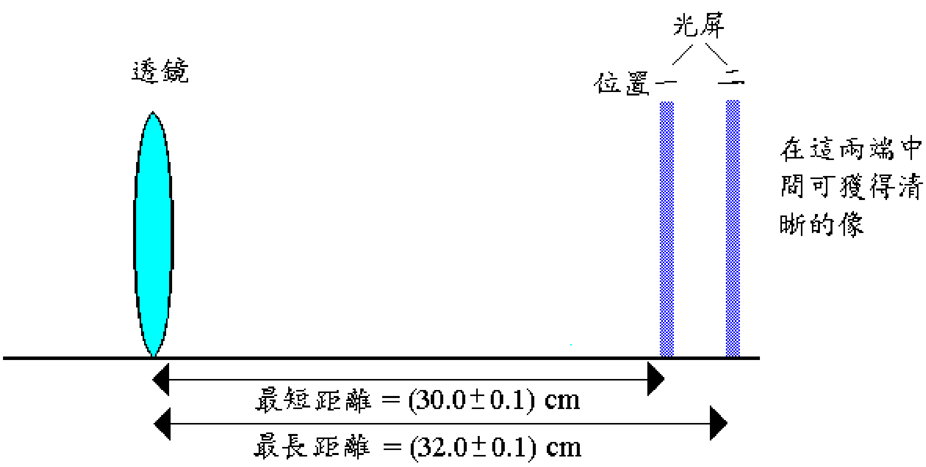
(甲) 絕對誤差、分數誤差和百分誤差

如某物理量的讀數是 x，真值是 X，則

1. 絕對誤差 $= \Delta X = |X - x|$
2. 分數誤差 $= \left|\dfrac{\Delta X}{X}\right| \approx \left|\dfrac{\Delta X}{x}\right|$
3. 百分誤差 $= \left|\dfrac{\Delta X}{X}\right| \cdot 100\% \approx \left|\dfrac{\Delta X}{x}\right| \cdot 100\%$

(乙) 在單一物理量中出現的誤差

在某些實驗中，可以用“依托技術”(bracketing techniques) 來量度某些值【例如：在光學實驗中找出像的位置】。以圖七為例，如在位置一至位置二之間放置屏幕，均可獲得清晰的像，像的正確位置很可能在距離透鏡 31 cm 處，其誤差為 1 cm。



圖七　量度像距

如以米尺量度 31.0 cm 的距離 v，則 $v = (31.0 + 0.1)$ cm 因米尺的最小標度是 0.1 cm。

找出像的位置時所牽涉的誤差 (1 cm)，比從標度讀取數據時的<u>不確定量</u> (0.1 cm) 為大，所以最大誤差應取 1 cm。
像距 $v = (31 \pm 1)$ cm。

例九　在一部光譜儀的轉台上，放置一片衍射光柵以量度鈉光的波長。量度一次所得的值為 (588±1) nm，為減小隨機誤差，進行多次量度，得出以下結果：

λ/nm	587	589	588	591	588	587	589	590	592	590

$$\overline{\lambda} = \frac{\sum_{i=1}^{10} \lambda_i}{10}$$

$$= \frac{587 + 589 + 588 + 591 + 588 + 587 + 589 + 590 + 592 + 590}{10}$$

$$= \cancel{589.1}$$

$$= 589 \text{ nm}$$

由於結果精確至最近的納米(nm)，故不應包含小數位後的數字

樣本標準差　$\Delta\lambda = 2$ nm

以上十個數據的平均值為 589 nm。樣本標準差 (2 nm) 比標度的不確定量 (1 nm) 為大。因此鈉光的波長應寫作 (589 ± 2) nm。

(丙)　誤差的組合

1.　和與差

$Z = A + B$　或　$Z = A - B$　其中 A 和 B 是獨立變數
為求簡易，可用個別的絕對誤差的總和表示 Z 的不確定量 ΔZ[*]。

即 $\Delta Z = |\Delta A| + |\Delta B|$，在這裏 ΔZ 實為 Z 的最大誤差。

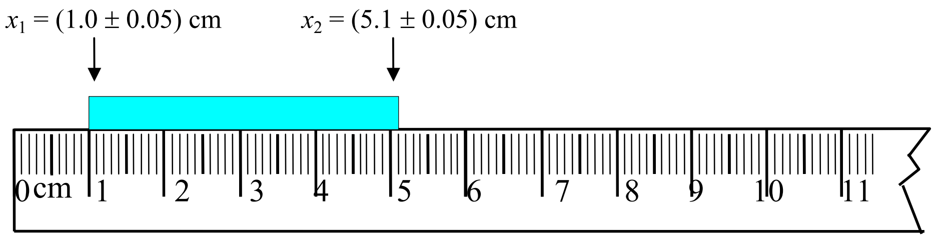
例十　若 $A = (76 \pm 3)$ cm 減去 $B = (15 \pm 2)$ cm，則
$Z = A - B = 76 - 15 = 61$ cm
而最大誤差
$\Delta Z = 3 + 2 = 5$ cm　即　$Z = (61 \pm 5)$ cm

[*] 由於個別誤差是獨立無關的，所以兩個誤差很少會同時有最大的正數值或最大的負數值。因此兩個誤差可能部分抵消。故 Z 的誤差的最佳估計值 (ΔZ_S) 應是兩個個別誤差的平方和的平方根：

$$\Delta Z_s = \sqrt{|\Delta A|^2 + |\Delta B|^2}$$

可是，為求簡易，建議同學使用最大誤差。

例十一　用米尺量度一件物體的長度。



物體長度　$= x_2 - x_1 = 5.1 - 1.0$

$= 4.1$ cm

而最大的誤差　$= 0.05 + 0.05$ cm $= 0.1$ cm

即物體長度　$= (4.1 \pm 0.1)$ cm

【註：使用米尺的中段來量度物體的長度，可避免因端點標度褪色或磨損而引致的零點誤差。】

例十二　(譯自　1989年　香港高級程度會考　物理科　卷一　第1題)

使用螺旋測微計來量度一條線的直徑，得出以下的讀數：

平均零點讀數　　-0.05 ± 0.02 mm　及

平均外觀直徑　　$+1.05 \pm 0.02$ mm

線的直徑應寫作

A. 1.00 ± 0.02 mm

B. 1.00 ± 0.04 mm

C. 1.10 ± 0.00 mm

D. 1.10 ± 0.02 mm

E. 1.10 ± 0.04 mm　　　　(答案: E)

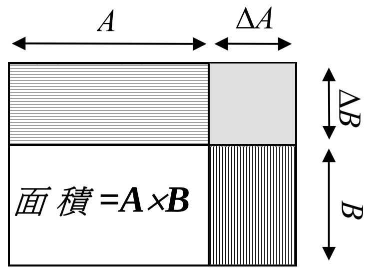
2. 積與商

$Z = A \cdot B$　或　$Z = \dfrac{A}{B}$　　　其中 A 和 B 是獨立變數

計算 Z 的不確定量時，我們採用分數誤差或百分誤差。Z 最大的分數誤差 $(\Delta Z/Z)$ 相當於 A 的分數誤差 $(\Delta A/A)$ 與 B 的分數誤差 $(\Delta B/B)$ 的總和:

$$\left|\frac{\Delta Z}{Z}\right| = \left|\frac{\Delta A}{A}\right| + \left|\frac{\Delta B}{B}\right|$$

我們可用代數方法來解釋乘法中的複合誤差的計算方法，假如某塊面積的真實長度爲 A 和真實闊度爲 B, 長和闊分別被量度成 $A + \Delta A$ 和 $B + \Delta B$, ，其中 ΔA 和 ΔB 爲絕對誤差。



計算所得的面積 $= (A + \Delta A) \cdot (B + \Delta B) = A \cdot B + A \cdot \Delta B + B \cdot \Delta A + \Delta A \cdot \Delta B$

真實面積 $= A \cdot B$

面積的誤差 = 計算所得的面積 − 真實面積

$$= A \cdot \Delta B + B \cdot \Delta A + \Delta A \cdot \Delta B$$

面積的最大誤差 $= |A \cdot \Delta B| + |B \cdot \Delta A| + |\Delta A \cdot \Delta B|$

面積的分數誤差

$$= \frac{|A \cdot \Delta B| + |B \cdot \Delta A| + |\Delta A \cdot \Delta B|}{A \cdot B}$$

$$= \left|\frac{\Delta B}{B}\right| + \left|\frac{\Delta A}{A}\right| + \left|\frac{\Delta A \cdot \Delta B}{A \cdot B}\right|$$

$$\approx \left|\frac{\Delta A}{A}\right| + \left|\frac{\Delta B}{B}\right|$$

$= A$ 的分數誤差 ＋ B 的分數誤差

例十三 在由量度速率和頻率得出聲音波長的計算中($v = f\lambda$)，

$v = (330 \pm 20)$ m s^{-1}, $\Delta v / v = 20/330 = 0.061$ (即 6.1 %)
$f = (512 \pm 10)$ Hz , $\Delta f / f = 10/512 = 0.020$ (即 2.0%)

則 $\lambda = 330/512 = 0.645$ m (三位有效數字)
λ 的最大分數誤差爲
$\Delta\lambda / \lambda = 0.061 + 0.020 = 0.081$ (即 8.1%), 及
$\Delta\lambda = 0.645 \times 0.081 = 0.052$ m

因此，由以上的數據所計算出的波長是
$\lambda = (0.65 \pm 0.05)$ m

3．冪

$Z = k \cdot A^n$　其中 k 和 n 是常數，並無誤差

Z 的分數誤差是 A 的分數誤差的 n 倍：

$$\left|\frac{\Delta Z}{Z}\right| = n \cdot \left|\frac{\Delta A}{A}\right|$$

例十四　(1993年　香港高級程度會考　物理科　卷一　第50題)

一平行板電容器由兩塊正方形的金屬片組成，爲求得電容量 C，一學生量度板的邊長 l 和兩板的間距 d。

若 l 的最大百分誤差 = 5%　及 d 的最大百分誤差 = 3%，C的最大百分誤差爲

A. 7%
B. 8%
C. 13%
D. 22%
E. 28%　(答案: C)

例十五　(1995年　香港高級程度會考　物理科　卷二　第45題)

在一個利用鋼球來量度鋼的密度的實驗中，所得數據如下：
鋼球的質量 = (530 ± 1) mg
鋼球的直徑 = (0.51 ± 0.01) cm

試求鋼的密度的計算值的百分誤差

A. 1%
B. 2%
C. 4%
D. 6%
E. 8%　(答案: D)

例十六　一個物體以速度 v 移動，若其質量 m = (3.5 ± 0.1) kg，速度 $v = (20 \pm 1)\ \text{m s}^{-1}$，求其動能的最大誤差。

$$E = \frac{1}{2}mv^2 = \frac{1}{2}(3.5)\cdot(20)^2 = 700\ \text{J}$$

$$\left|\frac{\Delta E}{E}\right| = \left|\frac{\Delta m}{m}\right| + 2 \times \left|\frac{\Delta v}{v}\right| = \frac{0.1}{3.5} + \frac{2 \times 1}{20} = \frac{9}{70}$$

即 $\Delta E = 700 \times \dfrac{9}{70} = 90\ \text{J}$

$E = (700 \pm 90)\ \text{J}$

例十七 (譯自 1990年 香港高級程度會考 物理科 卷一 第1題)

用公式 $T^2 = \frac{4\pi^2 l}{g}$ 計算重力加速度 g。
若 l 的最大百分誤差＝2%
T 的最大百分誤差＝5%，
g 的最大百分誤差爲

A. 3%
B. 8%
C. 12%
D. 23%
E. 27% (答案: C)

例十八 (1992 香港高級程度會考 物理科 卷一 第50題)

爲求出金屬線的橫截面積，一學生量度其直徑，所得數值爲 0.20 mm，誤差爲 ±0.02 mm。下列哪一項最能表達所得的結果？

A. 0.03 ± 0.01 mm^2
B. 0.031 ± 0.003 mm^2
C. 0.031 ± 0.006 mm^2
D. 0.0314 ± 0.0031 mm^2
E. 0.0314 ± 0.0063 mm^2 (答案: C)

例十九 (1996年 香港高級程度會考 物理科 卷二 第45題)

單擺的擺動周期 T 與長度 l 的關係可由公式 $T = 2\pi\sqrt{\frac{l}{g}}$ 表示。某學生利用一單擺實驗，以求出自由落體的加速度 g，如該學生獲得以下數據

擺動15次的時間 ＝ 14.4 ± 0.2 s
單擺的長度 ＝ 0.229 ± 0.001 m

以下何者最能合適地表示出加速度 g？

A. 9.8 ± 0.2 m s^{-2}
B. 9.8 ± 0.3 m s^{-2}
C. 9.81 ± 0.32 m s^{-2}
D. 9.81 ± 0.179 m s^{-2}
E. 9.81 ± 0.315 m s^{-2} (答案: B)

4. 其他組合

方程式中可能出現某些函數如正弦、餘弦和對數。計算其誤差，較前述的情況複雜。最簡易的做法是先將讀數的平均值，代入方程式求值。然後將最大和最小的讀數代入方程式，以取得兩極端值，再由此推算最大誤差。

例二十 在某光學實驗中，可由公式 $\lambda = d \sin \theta$ 求得單色光的波長λ，若 $d = (3.30 \pm 0.05) \times 10^{-6}$ m，而量度得的 $\theta = 10.1°, 9.9°, 10.3°$ 和 $10.2°$，求 λ 的數值及其最大誤差。

$$\begin{aligned}
\overline{\lambda} &= d \times \sin\overline{\theta} \\
&= 3.30 \times 10^{-6} \times \sin\left(\frac{10.1^{\circ} + 10.3^{\circ} + 10.2^{\circ} + 9.9^{\circ}}{4}\right) \\
&= 3.30 \times 10^{-6} \times \sin(10.125^{\circ}) \\
&= 5.80 \times 10^{-7} \text{ m}
\end{aligned}$$

因 $\sin\theta$ 是遞增函數，故 λ 的兩極端值為

$$\lambda_{min} = d_{min} \times \sin\theta_{min} = 3.25 \times 10^{-6} \times \sin(9.9^{\circ}) = 5.59 \times 10^{-7} \text{ m}$$

$$\lambda_{max} = d_{max} \times \sin\theta_{max} = 3.35 \times 10^{-6} \times \sin(10.3^{\circ}) = 5.99 \times 10^{-7} \text{ m}$$

$$\lambda_{max} - \overline{\lambda} = 0.19 \times 10^{-7} \text{m}$$

$$\overline{\lambda} - \lambda_{min} = 0.21 \times 10^{-7} \text{m}$$

$$\therefore \text{最大誤差 } \Delta\lambda_{\max} = 0.21 \times 10^{-7} \text{m}$$

即 $\overline{\lambda} = (5.8 \pm 0.2) \times 10^{-7}$ m

六、香港高級程度會考例子

(1997年 香港高級程度會考 物理科 卷二 第45題)
在量度一單擺的振盪週期 T 的實驗中，錄取得若干個全振盪的時間 t。結果測得30個全振盪的時間爲 28.7 ± 0.3 s。下列哪些敍述正確？

(1) 藉數算50個全振盪可減少 t 的讀數誤差。
(2) T 的百分誤差與 t 的相同。
(3) 所量度得的週期 T 應寫成 0.96 ± 0.01 s。

A. 只有 (1)
B. 只有 (3)
C. 只有 (1) 和 (2)
D. 只有 (2) 和 (3)
E. 只有 (1)、(2) 和 (3) (答案: D)

(1999年 香港高級程度會考 物理科 卷二 第45題)
下列哪些實驗量度會引致系統誤差？

(1) 使用運行較慢的秒表測量時間。
(2) 作輻射量度時，略去本底輻射讀數。
(3) 在光學實驗中，測不準最清晰成像的位置。

A. 只有 (1)
B. 只有 (3)
C. 只有 (1) 和 (2)
D. 只有 (2) 和 (3)
E. (1)、(2) 和 (3) (答案: C)

(2001年 香港高級程度會考 物理科 卷二 第36題)
在測定鋼線的楊氏模量的實驗中，一學生得到以下數據：

鋼線的長度	=	1.96 ± 0.01 m
鋼線的直徑	=	0.61 ± 0.01 mm
負載質量	=	10.00 ± 0.01 kg
伸長量	=	3.9 ± 0.1 mm
自由落體加速度	=	9.8 ± 0.1 m s^{-2}

計算楊氏模量時，下列哪項會引致最大的誤差？

A. 長度的量度
B. 直徑的量度
C. 負載質量的量度
D. 伸長量的量度
E. 自由落體加速度的假設值 (答案: B)

(2002 年 香港高級程度會考 物理科 卷二 第 45 題)
以小量水銀注入毛細管可測量管膛的直徑。水銀段的長度可量度至 2% 的準確度，而所用水銀的質量可量度至 4% 的準確度。假定水銀密度的誤差可略去，所計算出毛細管膛徑的百分誤差最多爲

A. 2%。
B. 3%。
C. 4%。
D. 6%。 (答案: B)

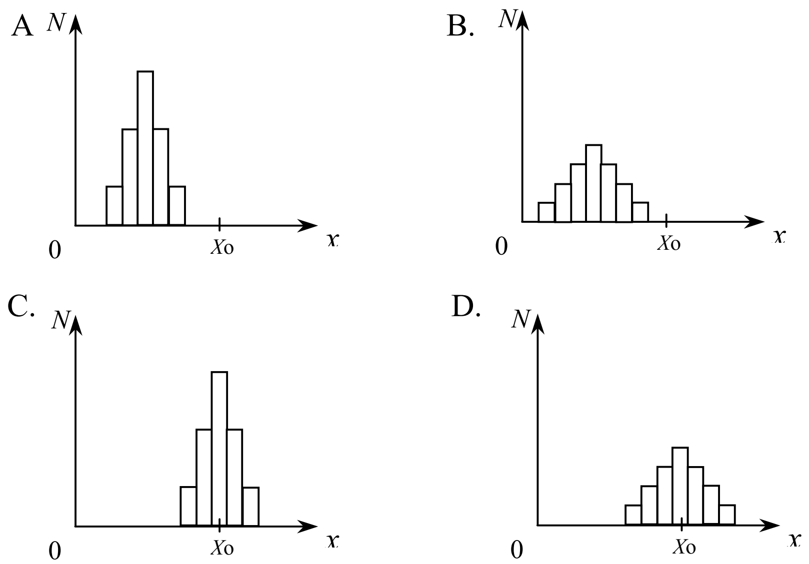
(2003 年 香港高級程度會考 物理科 卷二 第 13 題)
P、Q、R 和 S 四位學生，分別利用單擺方法量度自由落體加速度 g，並以數次量度所得的值求平均數。結果表列如下：

學生	結果 g/m s^{-2}
P	9.4 ± 0.8
Q	9.6 ± 0.2
R	9.8 ± 0.3
S	9.9 ± 1.2

哪位學生得到的結果有最大的系統誤差？

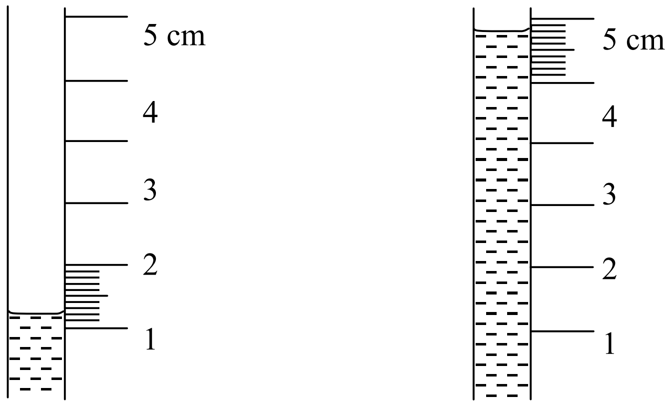
A. S
B. R
C. Q
D. P (答案: D)

(2005 年 香港高級程度會考 物理科 卷二 第 25 題)
某物理量 x 被多次測量。如果量得某 x 值的次數爲 N，並將 N 值對應 x 值繪圖。設該物理量的「真」值爲 x_o，以下圖中，哪一個顯示所作量度的隨機誤差小而系統誤差大？ (答案: A)



(1999 年 香港高級程度會考 物理科 卷一 第 1 題)

一位學生要量度鋼的密度。他把 20 粒相同的軸承鋼珠放入盛水半滿的量筒內。圖 1.1 顯示放入鋼珠前後，量筒內水面的讀數。所用的尺子上附有毫米刻度。



未加鋼珠　　　　已加鋼珠

圖 1.1

(a) 錄得水面升高的值爲 ______ ± ______ (單位：____________)。

(b) 將鋼珠放入量筒時，舉出一點要注意的事項，並加解釋。

(c) 已測得量筒的內直徑爲 9.40 + 0.05 mm，鋼珠的總質量爲 25.00 + 0.01 g。

- (i) 寫出量度量筒內直徑的儀器的名稱。
- (ii) 試求鋼珠的密度。
- (iii) 在 (ii) 中所算得的值，其百分誤差是多少？列出各運算步驟。

(d) 試述另一個能以較少百分誤差求得鋼珠密度的方法，並簡單解釋。

(2003 年 香港高級程度會考 物理科 卷一 第 9 題)

一位學生爲求一卷均勻漁絲的密度和楊氏模量，進行以下的實驗。

(a) 這卷漁絲的質量以電子秤量度。把漁絲放入盛水的量筒可求得其體積。所得結果如下 :

漁絲質量 $m = 2.756$ g

初始水平讀數 $l_1 = 5.0$ ml

最終水平讀數 $l_2 = 7.6$ ml

(i) 舉出量度漁絲體積時一點要注意的事項。

(ii) 計算這卷漁絲的密度，準確至 2 位有效數字。

(iii) 電子秤和量筒的最小分格分別爲 0.001 g 和 0.1 ml。估算用這個方法求取 時的最大百分誤差。

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